HL Math Complex Numbers: Study Guide for IB May 2004

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Contents

1 What Are They?

1.1 Extension of Number System

A few times in your life already, you have extended the set of numbers that you have been able to work with.

$$\mathbb{N} \quad \rightarrow \quad \mathbb{Z} \quad \rightarrow \quad \mathbb{Q} \quad \rightarrow \quad \mathbb{R} \quad \rightarrow \quad \mathbb{C}$$

e.g.

$$\begin{aligned}
&x^2 + 2x + 3 = 0 \\
\text{so far:} \quad & b^2 - 4ac = 4 - 4 \cdot 1 \cdot 3 = -8 \\
& \text{No Real Solution} \\
\text{But now:} \quad x &= \frac{-2 \pm \sqrt{-8}}{2 \cdot 1} \\
&= \frac{-2 \pm \sqrt{8}\sqrt{-1}}{2} \\
&= \frac{-2 \pm 2\sqrt{2}i}{2} \\
&= -1 \pm \sqrt{2}i \\
& \quad \Downarrow \qquad \Downarrow \\
& \text{Real} \quad \text{Imaginary}
\end{aligned}$$

1.2 $\Re$ and $\Im$

We define two operations:

$$\begin{aligned}
\text{if} \quad z &= a + bi \\
&\Downarrow \\
\Re(z) &= a \\
\Im(z) &= b \qquad \text{or } bi
\end{aligned}$$

1.3 Complex Conjugate

We define the complex conjugate of z as:

$$\begin{aligned}
\text{If} \quad z &= a + bi \\
&\Downarrow \\
\bar{z} &= a - bi
\end{aligned}$$

2 Basic Arithmetic

e.g.

$$z = 2 + 3i \qquad w = 4 - i$$

2.1 Addition and Subtraction

$$\begin{aligned} z + w &= (2 + 3i)(4 - 1) \\ &= 6 + 2i \\ z - w &= -2 + 4i \end{aligned}$$

2.2 Multiplication

$$\begin{aligned} z \cdot w &= (8 + 3) + (-2 + 12)\, i \\ &= 11 + 10i \end{aligned}$$

2.3 Division

$$\begin{aligned} \frac{z}{w} &= \frac{2 + 3i}{4 - 1} \\ &= \frac{2 + 3i}{4 - 1} \cdot \frac{4 + i}{4 + i} \\ &= \frac{5 + 14i}{17} \\ &= \frac{5}{17} + \frac{14}{17} i \end{aligned}$$

2.4 Simultaneous Equations

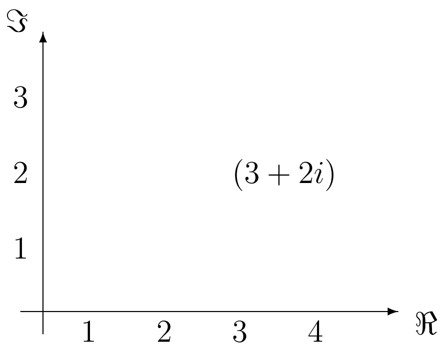
$$\begin{aligned} z + iw &= i + 1 && (1) \\ 2z - w &= i && (2) \\ 2(1) - (2) & \\ 2iw + w &= 2(1 + i) - i \\ w(2i + 1) &= 2 + i \\ w &= \frac{2 + i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} \\ &= \frac{4}{5} - \frac{3}{5} i \end{aligned}$$

2.5 Square Roots

$$\begin{aligned}
\sqrt{3+4i} &= a+bi \quad \cdot 2 \\
3+4i &= (a+bi)^2 \\
&= (a^2-b^2)+2abi \\
\text{Comparing } \Re \text{ and } \Im \text{ parts:} \\
&\Downarrow \\
a^2+b^2 &= 3 \\
2ab &= 4 \qquad \rightarrow b = \frac{2}{a} \\
&\Downarrow \\
a^2 - \frac{4}{a^2} &= 3 \\
\left(a^2\right)^2 = 3a^2 - 4 &= 0 \\
\left(a^2-4\right)\left(a^2+1\right) &= 0 \\
&\Downarrow \\
a = 2 \quad &\rightarrow \quad b = 1 \\
a = -2 \quad &\rightarrow \quad b = -1 \\
\sqrt{3+4i} &= 2+i \\
&= -2-i \\
&= \pm(2+i)
\end{aligned}$$

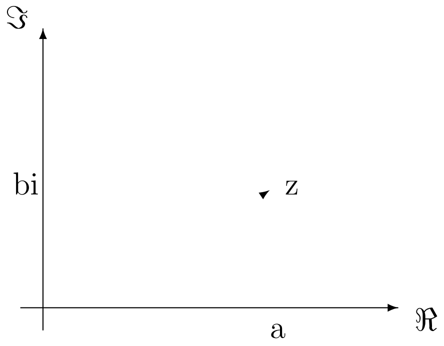
3 The Geometrical Representation

3.1 Argand Diagram



4 The Modulus-Argument Form

4.1 Definition



$$\begin{aligned} & z = a + bi \\ \text{Modulus} \quad & r = |z| \\ & = \sqrt{a^2+b^2} \qquad r \geq 0 \end{aligned}$$

$$\text{Argument} \quad \tan\theta = \frac{b}{a} \qquad 0^\circ \leq \theta \leq 360^\circ$$

4.2 Euler's Identity

$$\begin{aligned} \text{Let:} \quad z &= \cos\theta + i\sin\theta \\ \frac{dz}{d\theta} &= -\sin\theta + i\cos\theta \\ &= iz \\ & \text{This is a differential equation,} \\ & \text{so separate variables} \\ \int \frac{1}{z}\,dz &= \int i\,dz \\ \ln z &= i\theta + c \\ z &= e^{i\theta+c} \\ &= e^{i\theta}\cdot e^{c} \\ & \text{Find } e^c: \text{ put } \theta = 0 \\ z &= \cos 0 + i\sin 0 \\ &= 1 \\ z &= e^c \\ & \Downarrow \\ e^c &= 1 \qquad (c=0) \\ z &= e^{i\theta} \\ \cos\theta + i\sin\theta &= e^{i\theta} \end{aligned}$$

4.3 Properties

4.3.1 Modulus

$$|wz| = |w| \cdot |z|$$
$$|\frac{w}{z}| = \frac{|w|}{|z|}$$

4.3.2 Argument

$$\arg wz = \arg w + \arg z$$
$$\arg \frac{w}{z} = \arg w - \arg z$$

4.4 deMoivre's Theorem

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n \\ &= e^{i(n\theta)} \\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

e.g. Find z^4

$$\begin{aligned}z &= \sqrt{2}e^{i\frac{\pi}{4}} \\ z^4 &= \sqrt{2}^4 \cdot e^{i\cdot 4\cdot\frac{\pi}{4}} \\ &= 4 \cdot e^{i\pi} \\ &= -4\end{aligned}$$

e.g.

$$\begin{aligned}\sqrt[3]{i} &= \left(1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right)^{\frac{1}{3}} \\ &\qquad + k2\pi \qquad + k2\pi \qquad \text{for } k = 0, 1, 2 \\ &= 1^{\frac{1}{3}} \cdot \left(\cos\left(\frac{\pi}{6} + k\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + k\frac{2\pi}{3}\right)\right) \\ n = 0 \qquad &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ n = 1 \qquad &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\ n = 2 \qquad &= -i\end{aligned}$$

5 Polynomial Equations

The solutions of any polynomial equation with real coefficients come in complex conjugate pairs;

*e.g.*a cubic equation with real coefficient has roots $1 + 2i$ and 3. Write down the third root and find the equation.

$$1 \bar{+} 2i = 1 - 2i$$

$$\begin{aligned}(z - \ldots)(z - \ldots)(z - \ldots) &= 0\\ (z - (1 + 2i))(z - (1 - 2i))(z - 3) &= 0\\ (z^2 - 2z + 5)(z - 3) &= 0\\ z^3 - 5z^{1}1z - 15 &= 0\end{aligned}$$

6 Loci