HL Math Differential Equations: Study Guide for IB May 2004

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February 10, 2004

Contents

Differential Equations

1 Definition

Example of a differential equation:

$$\begin{array}{rrcl} & \frac{d^2y}{dx^2} & = & -y \quad (\text{meaning } y'' = -y) \\ \text{Solution:} & y & = & A\sin x + B\cos x \end{array}$$

Another Example:

$$\begin{array}{rrcl} & \frac{dy}{dx} & = & y \\ \text{Solution:} & y & = & Ae^x \end{array}$$

Where do they come from?
From physics, chemistry, economics, etc'.

e.g. an object is thrown upwards with a speed of 10m/s from a height of 30m. Write down the differential equation and solve it to find the equation of motion; (Reminder: velocity $\rightarrow \frac{dx}{dt}$, acceleration $\rightarrow \frac{d^2x}{dt^2}$)

$$\begin{array}{rrcl}
& \dfrac{d^2x}{dt^2} & = & -g \quad \text{when } g \approx 10 \\
& & = & -10 \\
& \dfrac{dx}{dt} & = & \int (-10)\, dt \\
& & = & -10t + c_1 \\
& \text{at } t & = & 10 \\
& & \Downarrow & \\
& \dfrac{dx}{dt} & = & -10t + 10 \\
& x & = & \int (-10t + 10)\, dt \\
& & = & -5t^2 + 10t + c_2 \\
& \text{at } t & = & 0 \\
& & \Downarrow & \\
& x & = & 30 \\
\text{Solution} \rightarrow & x & = & -5t^2 + 10t + 30
\end{array}$$

2 Solving Differential Equations: Separating Variables

e.g.

$$\begin{aligned}
\frac{dy}{dx} &= y \\
&\text{Separating Variables} \\
\int \frac{1}{y}\,dy &= \int dx \quad \rightarrow \text{(same as: } \int 1\,dx) \\
\ln y + c_1 &= x + c_2 \\
\ln y &= x + c \\
e^{\ln y} &= e^{x+c} \\
y &= e^x e^c \\
y &= Ce^x
\end{aligned}$$

3 Homogeneous Differential Equations

e.g.

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

Variables cannot be separated

$$\begin{aligned}
\text{But} \quad \frac{dy}{dx} &= \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \\
\text{Let} \quad u &= \frac{y}{x} \\
\text{So} \quad y &= ux \\
\frac{du}{dx}\cdot x &= \frac{u-1}{u+1} - u \\
&= \frac{u-1-u^2-u}{u+1} \\
&= -\frac{1+u^2}{1+u} \\
\int \frac{1+u}{1+u^2}\,du &= \int \frac{1}{x}\,dx
\end{aligned}$$