HL Math Integration: Study Guide for IB May 2004

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Contents

1 'Anti-Differentiation'

Integration can be seen as the inverse operation of differentiation. However, it only determines a function up to an additive constant.

$$\int x dx = \sin x + c$$

1.1 Some Rules

$$\begin{aligned}
\int \Big(f(x) + g(x)\Big) dx &= \int f(x)dx + \int g(x)dx \\
\int k \cdot f(x)dx &= k \cdot \int f(x)dx \\
\int x^n dx &= \frac{1}{n+1} x^{n+1} + c \\
\int \sin x dx &= \cos x + c \\
\int e^x &= e^x + c \\
\int x^{-1} &= \frac{1}{x} + c \quad (= \ln x + c \\
\int f(ax+b)dx &= \frac{1}{a} \cdot F(ax+b) + c
\end{aligned}$$

2 Definite Integral

The definite integral gives us the area between the limits, meaning:

if $y = x^2$ and $A =$ the area below the graph between $x = 1$and $x = 2$, then:

$$A(x) = \int x^2 dx = \frac{x^3}{3} + c$$

The usual way of writing this:

$$\begin{aligned}
\text{Area} &= \int_1^2 x^2 dx \quad \leftarrow \text{definite integral} \\
&= \left[\frac{x^3}{3}\right]_1^2 \\
&= \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right) \\
&= \frac{7}{3}
\end{aligned}$$

`Note: The definite intgeral of areas below the x-axis is negative:`

$$\int_{\frac{-\pi}{2}}^{0} \sin\theta \, d\theta = -1$$

3 Methods of Integration

It is important to note that:

1. finding the right method is not automatic but requires invention.
2. most integrals cannot be integrated, *e.g.*

$$\int e^{x^2} dx$$

3.1 Substitution

e.g.

$$\begin{aligned}
\frac{d}{dx}e^{x^2} &= e^{x^2} \cdot 2x \\
\int e^{x^2} \cdot 2x dx &= ? \\
\text{let } u &= x^2 \\
\frac{du}{dx} &= 2x \\
du &= 2x dx \\
&= \int e^u du \\
&= e^u + c \\
&= e^{x^2} + c
\end{aligned}$$

3.1.1 Patterns

$$\begin{aligned}
\int \frac{f'(x)}{f(x)} &= \ln f(x) + c \\
&\downarrow \\
e.g. \quad \frac{1}{3}\int \frac{3x^2}{x^3 - 1} dx &= \frac{1}{3}\ln\left(x^3 - 1\right) + c \\
\int f'(x) \cdot e^{f(x)} dx &= e^{f(x)} + c
\end{aligned}$$

3.2 Integration by Parts

$$\begin{aligned} y &= \int u'v\,dx \\ &= uv - \int uv'dx \end{aligned}$$

e.g.

$$\begin{aligned} y &= \int x \cdot e^{2x} dx \\ &= x \cdot \frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} dx \\ &= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c \\ &= \frac{1}{4} e^{2x}(2x-1) + c \end{aligned}$$

3.2.1 The 'neatest' Trick

$$\begin{aligned} y &= \int \frac{1}{x} \cdot \ln x\,dx \quad (= I) \\ &= \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x}\,dx \\ I &= (\ln x)^2 - I \\ 2I &= (\ln x)^2 \\ I &= \frac{1}{2}(\ln x)^2 + c \end{aligned}$$

4 Partial Fractions

4.1 Definition

e.g.

$$\begin{aligned} \frac{2}{x^2-1} &= \frac{2}{(x+1)(x-1)} \\ &= \frac{A}{x+1} + \frac{B}{x-1} \\ &= \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} \\ 2 &= A(x-1)+B(x+1), \quad \text{for all } x \end{aligned}$$

4.2 Methods and Shortcut

In order to find A and B:

1. Find A and B

$$\begin{array}{rclcrclrcl} x &=& 1: & 2 &=& 2B & B &=& 1 \\ x &=& -1: & 2 &=& -2A & A &=& -1 \\ \text{or } x &=& 3: & 2 &=& 2A+4B & A &=& -1 \\ & & & 2 &=& -A+B & B &=& 1 \end{array}$$

2. Compare Coefficients

$$\left.\begin{array}{lrcl} x: & 0 &=& A+B \\ 1: & 2 &=& -A+B \end{array}\right\} \begin{array}{rcl} B &=& 1 \\ A &=& -1 \end{array}$$

$$\text{hence,} \quad \tfrac{2}{x^2-1} = \tfrac{-1}{x+1} + \tfrac{1}{x-1}$$

3. The 'Cover-Up' Method

$$\begin{aligned} \tfrac{2}{x^2-1} &= \tfrac{2}{(x+1)(x-1)} \\ &= \tfrac{-1}{x+1} + \tfrac{1}{x-1} \end{aligned}$$

4.3 Quadratic Factor

e.g.

$$\begin{aligned} y &= \tfrac{2x+1}{(x-2)(x^2+1)} \\ &= \tfrac{A}{x-2} + \tfrac{Bx+C}{x^2+1} \\ &= \tfrac{A(x^2+1)+(Bx+C)(x-2)}{(x-2)(x^2+1)} \end{aligned}$$

$$2x+1 = A(x^2+1)+(Bx+C)(x-2) \qquad \text{for all x}$$

$$\begin{array}{rclcrclrcl} x &=& 2: & 5 &=& 5A & A &=& 1 \\ x &=& 0: & 1 &=& A-2C & C &=& 0 \end{array}$$

$$\begin{array}{lrcll} \text{Coeffecient of } x^2\text{:} & 0 &=& A+B & \rightarrow B=-1 \\ & & \Downarrow & & \\ & y &=& \tfrac{1}{x-2} - \tfrac{x}{x^2+1} & \end{array}$$

4.4 Repeated Factor

e.g.

$$\begin{aligned} y &= \tfrac{2x+1}{(x-2)(x+1)^2} \\ &= \tfrac{A}{x-2} + \tfrac{B}{x+1} + \tfrac{C}{(x+1)^2} \\ &= \tfrac{A(x+1)^2+B(x+1)(x-2)+C(x-2)}{(x-2)(x+1)^2} \end{aligned}$$

$$2x+1 = A(x+1)^2 + B(x+1)(x-2) + C(x-2) \qquad \text{for all x}$$

$$\begin{array}{rcrrclrcl} x &=& 2: & 5 &=& 9A & A &=& \frac{5}{9} \\ x &=& -1: & -11 &=& -3C & C &=& \frac{1}{3} \end{array}$$

$$\text{Coeffecient of } x^2\text{:} \qquad 0 = A + B \qquad \to B = -\tfrac{5}{9}$$

$$\Downarrow$$

$$y = \frac{\frac{5}{5}}{x-2} - \frac{-\frac{5}{9}}{x+1} + \frac{\frac{1}{3}}{(x+1)^2}$$

4.5 Degree of Top Greater than Degree at Bottom

In order to simplify a fraction with a higher degree (power) on the top than on the bottom, you should do polynomial division, which will simplify the fraction sufficiently so you can employ partial fraction methods.

5 Summary

Product:	Substitution	$\to$	$\int \cos x \cdot \sin^3 x \, dx$
	By Parts	$\to$	$\int x \cdot e^{2x} \, dx$
	Multiply Out	$\to$	$\int x(\sqrt{x} - 1) \, dx$
Quotient:	Divide Out	$\to$	$\int \frac{\sqrt{x}-1}{x} \, dx$
	Turn Into Product	$\to$	$\int \frac{\ln x}{x^3} \, dx$
	Partial Fractions	$\to$	$\int \frac{x+1}{x(x-1)} \, dx$
	Substitution	$\to$	$\int \frac{x}{x^2-3} \, dx$