HL Math Vectors:
Study Guide for IB May 2004

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Contents

1 The Basic Idea

A vector is an entity completely defined by a number, called the magnitude of the vector, and a direction; vectors are often usually represented by arrows (the length represents the magnitude):

Vectors	Scalars
displacement	temperature
velocity	speed
force	distance
	mass
	time

2 Vector Arithmetic

2.1 Multiplication by a Scalar

$\lambda\vec{a}$ is a vector whose magnitude is $|\lambda|$ times the magnitude of $\vec{a}$, and whose direction is the same $\vec{a}$ if $\lambda > 0$, and the opposite if $\lambda < 0$.
Special Case: a vector whose magnitude is 0 is called the zero vector, which is of no particular direction: $\vec{0}$

2.2 Vector Addition

One can use the cosine-rule to find out how far I am form the start. Either say:

- "follow one, then the other"
- "complete the parallelogram"

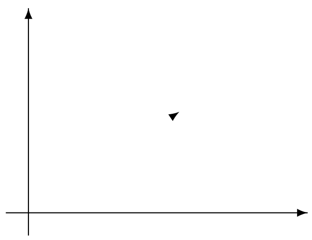
2.3 Subtracting Vectors

It is sensible to define $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Often it is useful to choose an origin O, relative to which other points can be fixed. The vector from O to the point P is called the position vector of P.

$\vec{XY} = \vec{OP} \quad \longrightarrow \quad$ but $\vec{XY}$ is not a position vector.

2.4 Position Vectors



We can switch between the two ways of representing vectors:

$$r = \sqrt{x^2 + y^2} \qquad x = r \cdot \cos\theta$$

or

$$\tan\theta = \frac{y}{x} \qquad y = r \cdot \sin\theta$$

The magnitude of a vector $\vec{a}$ is represented by $|\vec{a}|$;

2.5 Unit Vectors

A unit vector is a vector of magnitude 1; we denote unit vectors by $\hat{a}$ or $\dot{a}$.

e.g. find a unit vector in the direction of: $\vec{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$;

$$\begin{aligned} |\vec{a}| &= \sqrt{3^2 + (-4)^2} \\ \hat{a} &= \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} \end{aligned} \qquad \begin{aligned} |\vec{b}| &= \sqrt{(-2)^2 + 1^2 + 2^2} \\ \hat{b} &= \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -0.667 \\ 0.334 \\ 0.667 \end{pmatrix} \end{aligned}$$

3 Multiplying Vectors

3.1 Scalar Product

3.1.1 Definition

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

3.1.2 Special Case

$$\begin{aligned} \theta = 90^\circ : \quad \vec{a} \cdot \vec{b} &= 0 \\ \theta = 0^\circ : \quad \vec{a} \cdot \vec{b} &= |\vec{a}| \cdot |\vec{b}| \\ a^2 &= a \cdot |\vec{a}|^2 \end{aligned}$$

The scalar product is so called because it is scalar; it is also called the dot-product because it is written with a dot. *i.e.* not $\vec{a} \times \vec{b}$

3.1.3 Scalar Product in Components

e.g.

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} &= (p\vec{i} + q\vec{j})(r\vec{i} + s\vec{j}) \\ &= pr\vec{i}^2 + ps\vec{i}\vec{j} + qr\vec{i}\vec{j} + qs\vec{j}^2 \\ \text{but} \quad \vec{i}^2 &= \vec{i}^2 = 1 \\ \text{and} \quad \vec{i} \cdot \vec{j} &= \vec{j} \cdot \vec{i} = 0 \\ &= pr + qs \end{aligned}$$

3.2 Vector Product

3.2.1 Definition

The vector product (or cross product) $\vec{a} \times \vec{b}$ of two vectors is the vector which is perpendicular to $\vec{a}$ and to $\vec{b}$ (such that $\vec{a}$, $\vec{b}$, $\vec{a} \times \vec{b}$ form a right-hand system), and the magnitude $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta$

The vector product can only be defined in 3D-space; in 2D-space the problem that there is no direction perpendicular to $\vec{a}$ and $\vec{b}$, in general; in 4D-space and above, the problem is infinitely many directions perpendicular to two given vectors;

The cross product is equal to the area of the parallelogram, or twice the area of the triangle.

3.2.2 Special Cases

$$
\begin{aligned}
\vec{a} \times \vec{a} &= \vec{0} \\
\vec{b} \times \vec{a} &= -\vec{a} \times \vec{b} \quad \text{(anti-commutative)} \\
\vec{i} \times \vec{i} &= \ldots = \ldots = \vec{0} \\
\vec{i} \times \vec{j} &= \vec{k} \\
\vec{j} \times \vec{k} &= \vec{i} \\
\vec{i} \times \vec{k} &= -\vec{j} \\
\vec{k} \times \vec{i} &= \vec{j}
\end{aligned}
$$

3.2.3 Vector Product in Components

e.g.

$$
\begin{aligned}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} p \\ q \\ r \end{pmatrix} &= (a\vec{i} + b\vec{j} + c\vec{k}) \times (p\vec{i} + q\vec{j} + r\vec{k}) \\
&= aq\vec{k} - ar\vec{j} - bp\vec{k} + \ldots \\
&= \begin{pmatrix} br - cq \\ cp - ar \\ aq - bp \end{pmatrix}
\end{aligned}
$$

3.3 The Triple Product

3.3.1 Definition

$$
\vec{a} \times \vec{b} \cdot \vec{c} = (\vec{a} \times \vec{b}) \cdot \vec{c}
$$

$$\Downarrow$$

Volume of Parallelogram

4 3D-space Geometry

4.1 Lines

The slope gives direction of the line. The position of any point on the line can be written as:

$$\vec{r} = \vec{a} + \lambda\vec{d} \quad \text{(for some value of } \lambda, \text{where } \lambda \text{ is a scalar)}$$

We can now talk about lines in 3D-space.

4.1.1 Parallel Lines

$$\begin{aligned} \vec{r} &= \vec{a}_1 + \lambda\vec{d}_1 \\ \vec{r} &= \vec{a}_2 + \mu\vec{d}_2 \\ \vec{d}_1 &\parallel \vec{d}_2 \\ &\text{i.e.} \\ \vec{d}_1 &= k\vec{d}_2 \quad \text{for some } k \end{aligned}$$

4.1.2 Perpendicular Lines

$$\vec{d}_1 \cdot \vec{d}_2 = 0$$

4.1.3 Intersection of Two Lines

Simultaneous equations (eliminate $\vec{r}$):

$$\vec{a}_1 + \lambda\vec{d}_1 = \vec{a}_2 + \mu\vec{d}_2$$

If two lines are not parallel nor perpendicular and they don't meet or intersect, they are called **SKEW**.

4.2 Planes

4.2.1 Definition

There are three versions of the equation of the plane:

4.2.2 Parametric Form

$$\vec{r} = \vec{a} + \lambda\vec{d} + \mu\vec{e}$$

e.g. find an equation of V-plane through: $A(1, -1, 2),\ B(2, -1, 3),\ C(-1, 0, 1)$

$$\begin{aligned} \vec{r} &= 0\vec{A} + \lambda\vec{AB} + \mu\vec{AC} \\ &= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

4.2.3 Cartesian Form

e.g.

$$\begin{aligned} r &= \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix} \\ x &= 1 + 2\lambda - 3\mu \qquad (1) \\ y &= 1 + 2\lambda \qquad (2) \\ z &= -4 + 3\lambda - \mu \qquad (3) \end{aligned}$$

$$\begin{aligned} (1) - 3(3) &= 13 - 7\lambda \qquad (4) \\ 7(2) - 2(4) &= 7y + 2x - 6z = 3 \\ \text{Solution:} &= 7y + 2x - 6z = 33 \end{aligned}$$

A cartesian equation of a plane is in the form: $ax + by + cz = d$

4.2.4 Normal Form

if p lies on this plane, $\vec{AP}$ is in the direction of the plane,
so:

$$\begin{aligned}
\vec{AP} \cdot n &= 0 \\
\vec{AP} &= (\vec{r} - \vec{a}) \\
&\Downarrow \\
(\vec{r} - \vec{a}) \cdot n &= 0 \\
\vec{r}\vec{n} &= \vec{a}\vec{n} \\
&= d
\end{aligned}$$

e.g. a plane is perpendicular to the vector $\vec{n} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, and passes through $A(0, 2, -3)$; find the normal form of the equation.

Normal Form:

$$\begin{aligned}
\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} &= d \\
\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} &= d \\
-10 &= d \\
\vec{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} &= -10
\end{aligned}$$

Cartesian Form:
if the point $P(x, y, z)$
lies on the plane, then:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} &= -10 \\
x - 2y + 2z &= -10
\end{aligned}$$

Parametric Form:
Find two or more points on the plane:
$B(0, 0, 5) \quad C(-10, 1, 1)$

$$r = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -10 \\ 1 \\ 1 \end{pmatrix}$$

4.2.5 Switching

4.3 Intersections

4.3.1 Two Lines

$$\begin{aligned}
\vec{r}_1 &= \vec{a}_1 + \lambda \vec{d_1} \\
\vec{r}_2 &= \vec{a}_2 + \mu \vec{d_2}
\end{aligned}$$

$$\begin{aligned} \text{Solve:} \quad \vec{a}_1 + \lambda \vec{d}_1 &= \vec{a}_2 + \mu \vec{d}_2 \\ &\Downarrow \\ \lambda &= \dots \\ \mu &= \dots \end{aligned}$$

4.3.2 A Line and a Plane

$$\begin{aligned} \vec{r} &= \vec{a} + \lambda \vec{d} \\ \vec{r} \cdot \vec{u} &= d \\ (\vec{a} + \lambda \vec{d}) \cdot \vec{n} &= d \\ &\Downarrow \\ \lambda &= \dots \end{aligned}$$

Or

$$\begin{aligned} \frac{x-}{\dots} &= \frac{y-}{\dots} = \frac{z-}{\dots} \\ ax + by + cz &= d \\ &\Downarrow \\ x &= \dots \\ y &= \dots \\ z &= \dots \end{aligned}$$

4.3.3 Two Planes

e.g.

$$\begin{aligned} \vec{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} &= 5 \\ x + 2y - z &= 5 \end{aligned} \qquad \begin{aligned} \vec{r} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} &= -2 \\ y + 3z &= -2 \end{aligned}$$

$$\begin{array}{lllllllll} \text{Let} & y = 1: & z = -1 & , & x = 2 & \rightarrow & P(2, 1, -1) \\ \text{Let} & z = 0: & y = -2 & , & x = 9 & \rightarrow & Q(9, -2, 0) \\ & & & & & & \text{lie on both planes} \end{array}$$

Or: just find one point on both planes and find $\vec{d}$ by:

$$\vec{n}_1 \times \vec{n}_2$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix}$$

4.3.4 Three Planes

e.g.

$$\begin{aligned} x + 2y - z &= 5 \\ y + 5z &= -2 \\ 2x + y - 2z &= 0 \end{aligned}$$

Note: there may be no common points, i.e. two points parallel, points form a prism.

4.4 Distances

4.4.1 Two Points

e.g. $A(1,-1,3)$, $B(2,1,-1)$

$$\begin{aligned} AB &= \sqrt{(2-1)^2+(1-(-1))^2+(-1-3)^2} \\ &= \sqrt{21} \end{aligned}$$

4.4.2 A Point and a Plane

e.g. $A(1,-1,3)$, $\vec{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}7\\-3\\1\end{pmatrix}$
Vector from A to the line:

$$\begin{aligned} \vec{r} - \vec{OA} &= \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}7\\-3\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} \\ &= \begin{pmatrix}0\\3\\-2\end{pmatrix} + \lambda\begin{pmatrix}7\\-3\\1\end{pmatrix} \end{aligned}$$

The vector should be perpendicular to the line:

$$\begin{aligned} (\vec{r} - \vec{OA}) \cdot \vec{d} &= 0 \\ \left(\begin{pmatrix}0\\3\\-2\end{pmatrix} + \lambda\begin{pmatrix}7\\-3\\1\end{pmatrix}\right) \cdot \begin{pmatrix}7\\-3\\1\end{pmatrix} &= 0 \\ -11 + \lambda \cdot 59 &= 0 \\ \lambda &= -\frac{11}{59} \end{aligned}$$

Find $\vec{r}$ on the line for this λ to find F.
Find the distance between A and F.

4.4.3 A Point and a Plane

$$\begin{aligned}
\vec{r}\cdot\vec{n} &= p \\
\text{Line BF:} \quad \vec{r} &= \vec{OB} + \lambda\vec{n} \\
&\quad \text{F: simultaneous equations} \\
(\vec{OB} + \lambda\vec{n})\cdot\vec{n} &= p \\
&\Downarrow \\
\lambda &= \ldots \\
\text{F} &= (\ldots, \ldots, \ldots) \\
&\quad \text{Distance of B from the plane:} \\
\text{BF} &= \sqrt{(\ldots - \ldots)^2 + (\ldots - \ldots)^2 + (\ldots - \ldots)^2}
\end{aligned}$$

e.g. Find the distance of the origin from $2x+3y+4z = 5$

$$\begin{aligned}
2x + 3y + 4z = 5 &= \vec{r}\begin{pmatrix}2\\3\\4\end{pmatrix} = 5 \\
\perp \text{ Line:} \quad \vec{r} &= \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\4\end{pmatrix} \\
&\quad F\text{: Simultaneous equations} \\
\left(\begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\4\end{pmatrix}\right)\cdot\begin{pmatrix}2\\3\\4\end{pmatrix} &= 5 \\
0 + 29\lambda &= 5 \\
&\Downarrow \\
\lambda &= \frac{5}{29} \\
&\quad F\left(\frac{10}{29}, \frac{15}{29}, \frac{20}{29}\right) \\
&\quad \text{Distance of O from the plane } OF \\
&= \sqrt{\left(\frac{10}{29}\right)^2, \left(\frac{15}{29}\right)^2, \left(\frac{20}{29}\right)^2} \\
&= \frac{5\cdot\sqrt{29}}{29} \\
&= \frac{5}{\sqrt{29}}
\end{aligned}$$

4.4.4 Two Planes

In order for two planes to have a distance, they **have** to be parallel. There are two methods to find the distance:

1. take any point on one plane and find it's distance for another plane.
2. find the distance of both from the origin and subtract.

4.4.5 A Line from a Plane

In order for a line and a plane to have a distance, they **have** to be parallel to each other. In order to find the distance:

Take any point on the line and find it's distance from the plane.